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I. INTRODUCTION

In this note we study two rather general classes of access where both polling and random access ALOHA are special cases, The general conclusions drawn are that mixed polling/random access schemes give maximal throughput for high traffic or unbalanced sources. We also show that pure random access is almost "optimal" for low traffic and balanced sources. "Optimal" has to be taken with slight skepticism due to the inadequacies of the source models and in one case, the restriction to the total class of access schemes considered.

We look at two major cases A and B described below.

Case A

In this case we assume that the j^{th} terminal of N wishes to transmit with some known probability P_j. We are free to assign to each terminal some probability α_j which it will use to decide whether to transmit, if it has a packet. The objective is to maximize the throughput in the time slot under consideration.

It should be noted that this model does not account for blocking but does include polling as a possible strategy. It is "greedy" in the sense that it tries to maximize the "throughput" in the next slot.

Case B

In this case we have exactly n out of N terminals that wish to transmit.

Here we wish to choose the optimum strategy of the type that divides the set N into two groups and then gives each terminal in the group permission to transmit with probability α. This is a modified polling scheme and the question is how much does it buy as a function of n and N.

II. CASE A - OPTIMUM STRATEGY

We assume each terminal independently has a packet to transmit with probability P_j. We assume $P_i \geq P_j$ if $i \leq j$, for i,j from 1 to N. To each terminal we assign a probability α_j which it uses to decide whether to transmit if it indeed has a packet. Thus a packet is transmitted from each terminal with probability $\alpha_j P_j$. The average success rate, S, is

$$S = \left[\sum_{j=1}^{N} \frac{P_j\alpha_j}{(1-P_j\alpha_j)}\right] \prod_{\ell=1}^{N} (1-P_\ell\alpha_\ell) = \sum_{j=1}^{N} P_j\alpha_j \prod_{\substack{\ell=1 \\ \ell\neq j}}^{N} (1-P_\ell\alpha_\ell) \qquad (1)$$

To facilitate discussion we define a set K of $\{\alpha_j\}$ and an S_K, p_K and M_K as follows:

$$S_K = \left(\sum_{\alpha_j \varepsilon K} \frac{P_j\alpha_j}{1-P_j\alpha_j}\right) p_K \triangleq M_K \cdot p_K$$

$$p_K = \prod_{\alpha_j \varepsilon K} (1-P_j\alpha_j) \qquad (2)$$

We define set N as the set of all N α_j's. To find an optimal set of $\{\alpha_j\}$ we first compute $\partial S_N/\partial\alpha_j$ as follows:

$$\frac{\partial S_N}{\partial \alpha_j} = \frac{P_j}{1-\alpha_j P_j}\, p_N - \frac{P_j}{1-\alpha_j P_j} \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \frac{\alpha_\ell P_\ell}{1-\alpha_\ell P_\ell} \cdot p_N$$

$$= \frac{P_j p_N}{1-\alpha_j P_j}\left[1 - M_N + \frac{\alpha_j P_j}{1-\alpha_j P_j}\right] \tag{3}$$

We assume $M_N > 1$. Then we have $\left.\frac{\partial S_N}{\partial \alpha_j}\right|_{\alpha_j=0} < 0$

and $\frac{\partial S_N}{\partial \alpha_j}$ is a monotone increasing function of α_j. Thus the optimum α_j has $\alpha_j=0$ or 1 depending on which results in the maximum throughput. If $M_N > 1$ then $\frac{\partial S_N}{\partial \alpha_j} > 0$ for entire range of α_j and hence all $\alpha_j=1$. In this case

$$M_N = \sum_{j=1}^{N} \frac{P_j}{1-P_j} \quad ; \quad \alpha_j=1 \quad \forall j \tag{4}$$

It is an easy matter to check whether this condition holds. Thus the optimum system has a set with $\{\alpha_j=1\}$ and a complement with $\{\alpha_j=0\}$. It now remains to find the optimal subset. Suppose we have a set K and we add a new α_ℓ to K to give $K'=K+\{j\}$. Similarly $K=K'-\{j\}$. We can write $S_{K'}$, $p_{K'}$, in terms of S_K, p_K, as follows ($\alpha_j=1$):

$$S_{K'} = (1-P_j)S_K + P_j p_K$$
$$p_{K'} = (1-P_j)p_K \tag{5}$$

Thus

$$S_{K'} - S_K = P_j(p_K - S_K) \tag{6}$$

From (6) we see that we maximize the throughput difference by adding a new terminal K if $S_K < p_K$ and we choose the largest $\{j, P_j \max\} \notin K$. If $S_K > p_K$ then any additional terminal will reduce the throughput. We define an extremal set of terminals as one where no more terminals can be added without causing a drop in throughput and the removal of any single terminal will result in a reduced set that will allow the addition of a new terminal and increase throughput. This means that an extremal set $\tilde{K}$ has

$$\begin{aligned} &\text{i)}\ S_{\tilde{K}} > p_{\tilde{K}} \\ &\text{ii)}\ S_{\tilde{K}-\{j\}} < p_{\tilde{K}-\{j\}} \qquad \{j\} \varepsilon K \end{aligned} \tag{8}$$

We have an optimal extremal set, K_0, if we cannot replace any $\{j\} \varepsilon K_0$ by any $\{\ell\} \notin K_0$. This means that, using (6),

$$S_{K_0-\{j\}+\{\ell\}} - S_{K_0-\{j\}} = P_\ell \left[p_{K_0-\{j\}} - S_{K_0-\{j\}} \right]$$

and hence $$S_{K_0-\{j\}+\{\ell\}} - S_{K_0} = (P_\ell - P_j) \left[p_{K_0-\{j\}} - S_{K_0-\{j\}} \right]$$

Since $p_{K_0-\{j\}} - S_{K_0-\{j\}} > 0$ for all $\{j\} \varepsilon K_0$ we see that $S_{K_0-\{j\}+\{\ell\}} > S_{K_0}$ if and only if $P_\ell > P_j$. Thus an optimal point is characterized by $P_\ell \leq P_j$ for all $\{j\} \varepsilon K_0$ and $\{\ell\} \notin K_0$. Note this optimal is unique up to exchanges amongst $\{j\}$'s with equal P_j. Thus we find the optimal set by iterating through the P_j from $P_1 \geq P_2 \ldots \geq P_N$ until we no longer increase throughput. A simple iteration of (5) will suffice.

III. DISCUSSION - CASE A

This result is not too surprising in retrospect. We know that if $P_1=1$ then maximum through will occur if we let that (known) terminal transmit. This is just the extension that lets the heavy users into the channel. Unfortunately the light interactive users are discriminated against and hence have an inordinately long waiting time. It is possible to modify this by introducing a return r_j for each user. This could reflect priority and delay sensitivity. If we look at the return R_N and obtain $\partial R_N/\partial \alpha_j$ then we get roughly the same results as before with an optimal $\alpha_j=0,1$. The system considered here is random accessing through the P_j's and it just shows that assymmetry in the system access rates produce better results. Unfortunately, the simple formulation of the optimal rule does not seem to exist for arbitrary r_j's. Attempts to find one failed.

A question then arises as to when random access is good. The answer seems to be when the terminals are homogeneous and no terminal can dominate the channel. Case B explores this a little more.

IV. CASE B - A STRATEGY TRADE

In this case we assume that we have and know that exactly n of N terminals wish to transmit. We find the optimum strategy amongst the class that divides the N terminals into two sets with K in one and N-K in the other. We allow all terminals in the K-set to transmit with probability α and all those in the N-K set must remain silent. The objective is to find the (K,α) that maximizes the throughput for each (n,N). The following results are indicated by the analysis.

1) Numerical results indicate that the optimum scheme has $\alpha=1$ for

$$K = \left\lfloor \frac{N+1}{n} \right\rfloor$$

2) For $n/N<e^{-1}$, $K=N$ and $\alpha=\frac{1}{n}$ is virtually optimal.

3) For fixed n/N, N moderately large, the optimum α vs K seems to be independent N.

4) For fixed K, the maximum throughput (probability of success) is not very sensitive to the exact value of α.

To find the throughput S_K we note that if ℓ of the n active terminals are in the set K then the probability of success is $\ell\alpha(1-\alpha)^{\ell-1}$. The probability of there being ℓ of n in K is just $\binom{K}{\ell}\cdot\binom{N-K}{n-\ell}$.

There are $\binom{N}{n}$ total possible combinations and hence the probability of success with ℓ in K is

$$\frac{\binom{K}{\ell}\binom{N-K}{n-\ell}}{\binom{N}{n}}\cdot \ell\alpha(1-\alpha)^{\ell-1}$$

The most terminals that we can have in the set K is the $\min(n,K)$ and the least is the $\max(n-N+K,0)$. Thus the throughput S_K is just

$$S_K = \sum_{\ell=\max(n-N+K,1)}^{\min(n,K)} \frac{\binom{K}{\ell}\binom{N-K}{n-\ell}}{\binom{N}{n}}\ \ell\alpha(1-\alpha)^{\ell-1} \qquad (7a)$$

$$= \sum_{\ell=\max(n+k-N,1)}^{\min(n,K)} \frac{\binom{n}{\ell}\binom{N-n}{K-\ell}}{\binom{N}{K}}\ \ell\alpha(1-\alpha)^{\ell-1} \qquad (7b)$$

Note from 7a and 7b that S_K is symmetric in n and K. This allowed for certain simplifications in the evaluation of S_K. If we let α=1 then

$$S_{K|\alpha=1} = \frac{K\cdot\binom{N-K}{n-1}}{\binom{N}{n}} \tag{8}$$

$$S_{K+1|\alpha=1}-S_{K|\alpha=1} = \frac{(K+1)\binom{N-K-1}{n-1}}{\binom{N}{n}} - \frac{K\binom{N-K}{n-1}}{\binom{N}{n}}$$

$$= \frac{\binom{N-K-1}{n-1}}{\binom{N}{n}(N-K-n+1)}\left[(N+1)-n(K+1)\right] \tag{9}$$

Since $n\leq N$, the [ ] in (9) is positive up to some $\hat{K}$ whereupon $S_{\hat{K}+1|\alpha=1}$ decreases. Hence,

$$\frac{N+1}{n} - 1 < \hat{K} \leq \frac{N+1}{n}$$

$$\text{or} \quad \hat{K} = \left\lfloor \frac{N+1}{n} \right\rfloor \tag{10}$$

For $n>\frac{N+1}{2}$, $\hat{K}$=1 and $S_{\hat{K}}$=n/N. In this case we cannot do better by using $\alpha\neq 1$. For $\frac{N+1}{2}\geq n>\frac{N+1}{3}$, $\hat{K}$=2 and $S_{\hat{K}}=(n/N)\cdot\frac{2(N-n)}{(N-1)}$. This is better throughput than one obtains for K=N, α=1/n and n>4. The latter is optimal for slotted ALOHA type systems. For the above reasons the numerical results we restricted to $n/N\leq .4$. S_K in (7) was evaluated for N=10, 100, and 1000 and n/N=.1, .2, and .4 for virtually all K. In each case the optimal α was found. The results of S_K

vs. K are shown in Figures 1, 2 and 3 for n/N=.1, .2, and .4, respectively. Figures 4, 5, 6 show the corresponding optimal α. It should be noted that in all cases the optimum throughput occurred for K<N but for n/N=.1 and .2 the difference between K=N and K<N was negligible. Figures 1, 2, and 3 show that, except for small samples, the maximum throughput versus K is almost an invariant function of N. The indecisive nature of the curves of α for small K in Figures 4, 5, 6 is due to the broad range near maximum of α vs throughput.

This is shown in Figure 7 where we have plotted for S_K as a function of α for fixed K, n=20, N=100, K=5, K=10, K=100. Note the broad range for near optimal α.

V. DISCUSSION - CASE B

This case lends further credence to the statement that slotted ALOHA (K=N) does not maximize throughput for a diverse class of slotted access schemes. In all cases (n>1) the K<N, $\alpha \leq 1$ produced the maximal throughput. However, in those cases when n was "large" (n>5), and $N/n \leq .3$, the K=N, $\alpha = 1/n$ was virtually optimal. Thus the comparison of ALOHA with polling or mixed polling/random access schemes should not be made on throughput but rather issues such as simplicity or delay. Further, if the total traffic is large $n/N \geq .35$ (or .18 in the unslotted case), then random access results in a definite degradation of throughput performance.

VI. CONCLUSIONS

ALOHA-type random access results in near optimal throughputs when the data sources are roughly similar and the total traffic is less than e^{-1}.

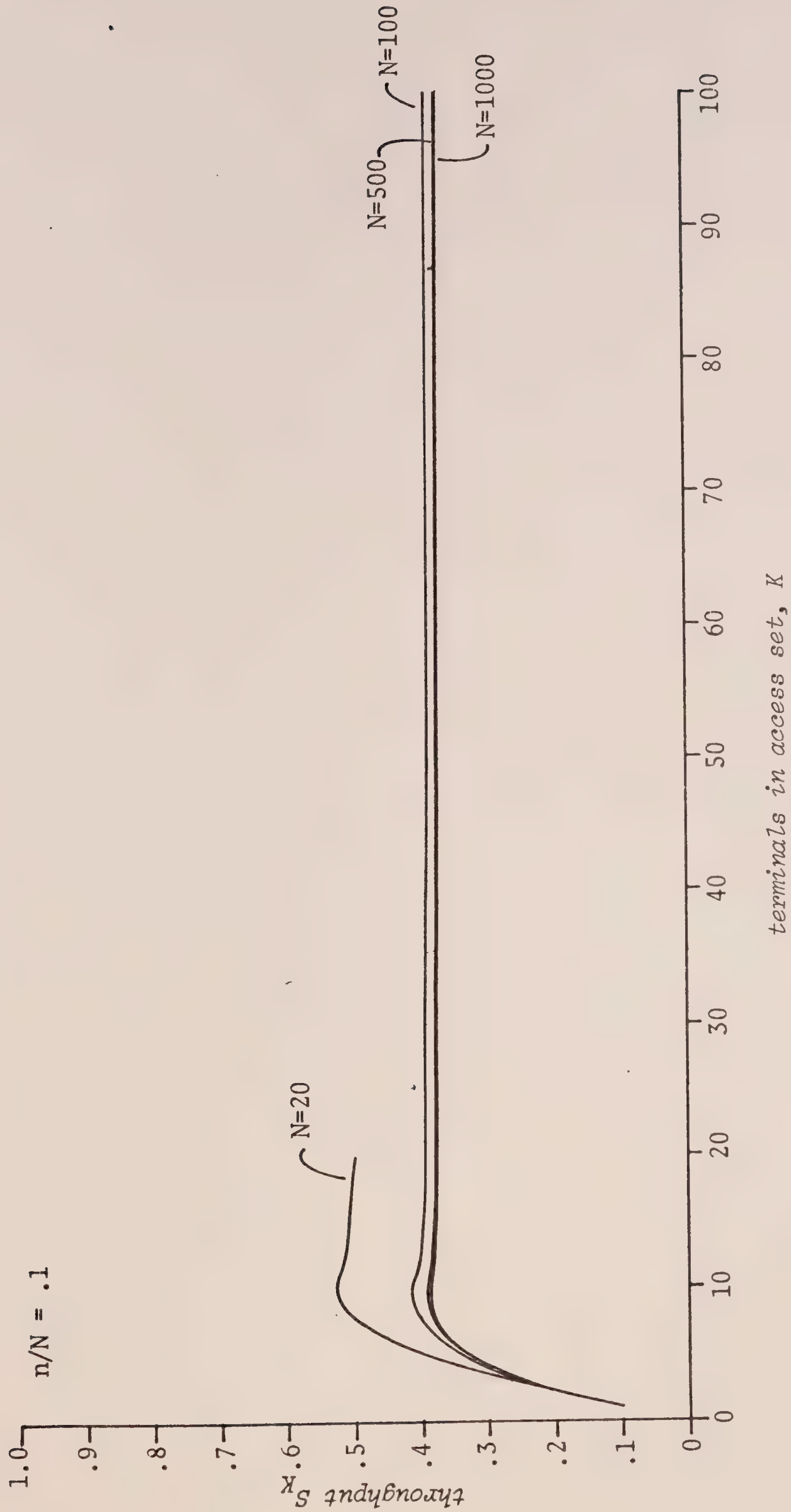
n/N = .1
N=20
N=100
N=500
N=1000
throughput S_K
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
10
20
30
40
50
60
70
80
90
100
terminals in access set, K

Figure 1

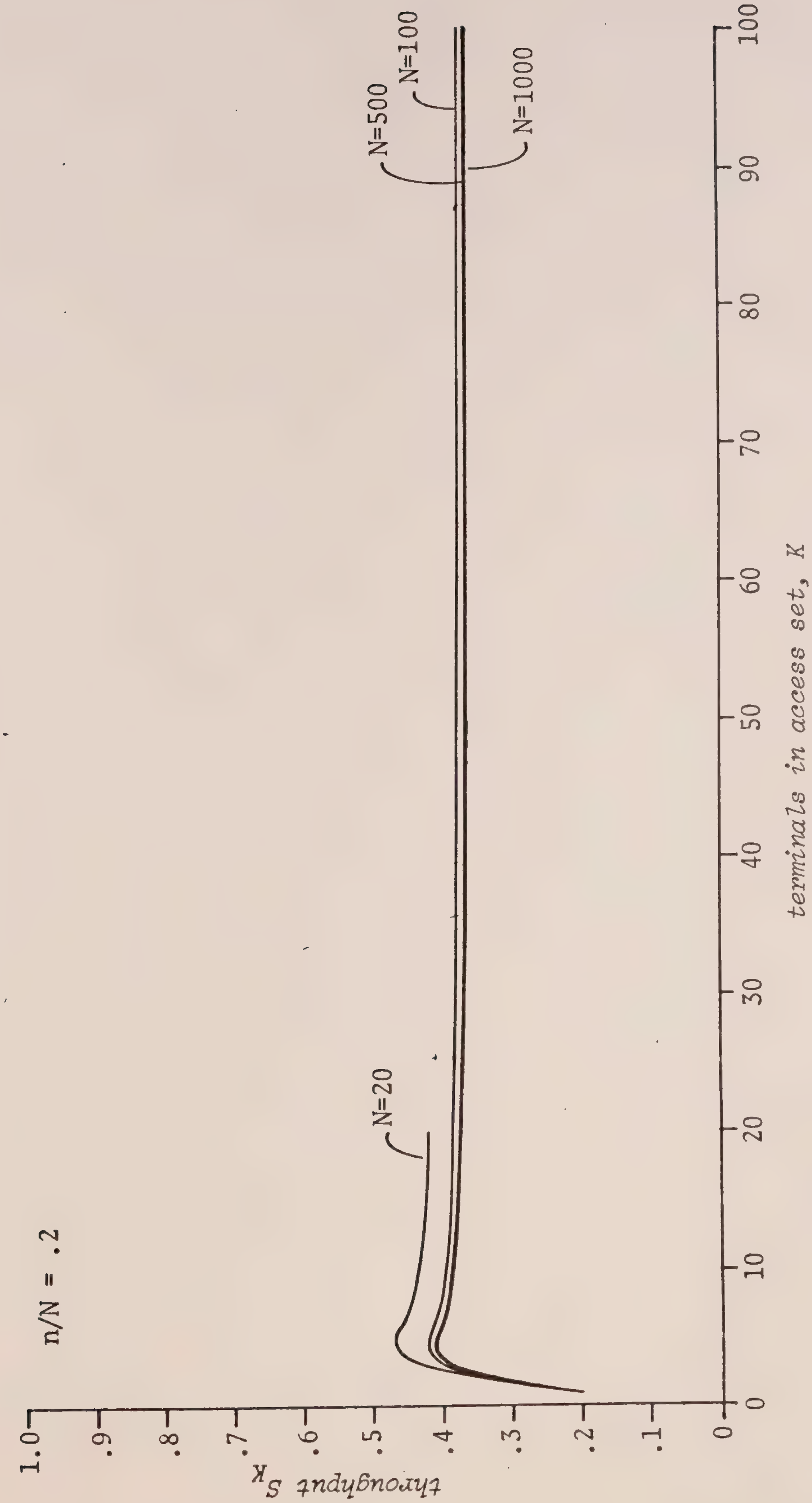
n/N = .2
N=20
N=500
N=100
N=1000
throughput S_K
terminals in access set, K
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
0
10
20
30
40
50
60
70
80
90
100

Figure 2

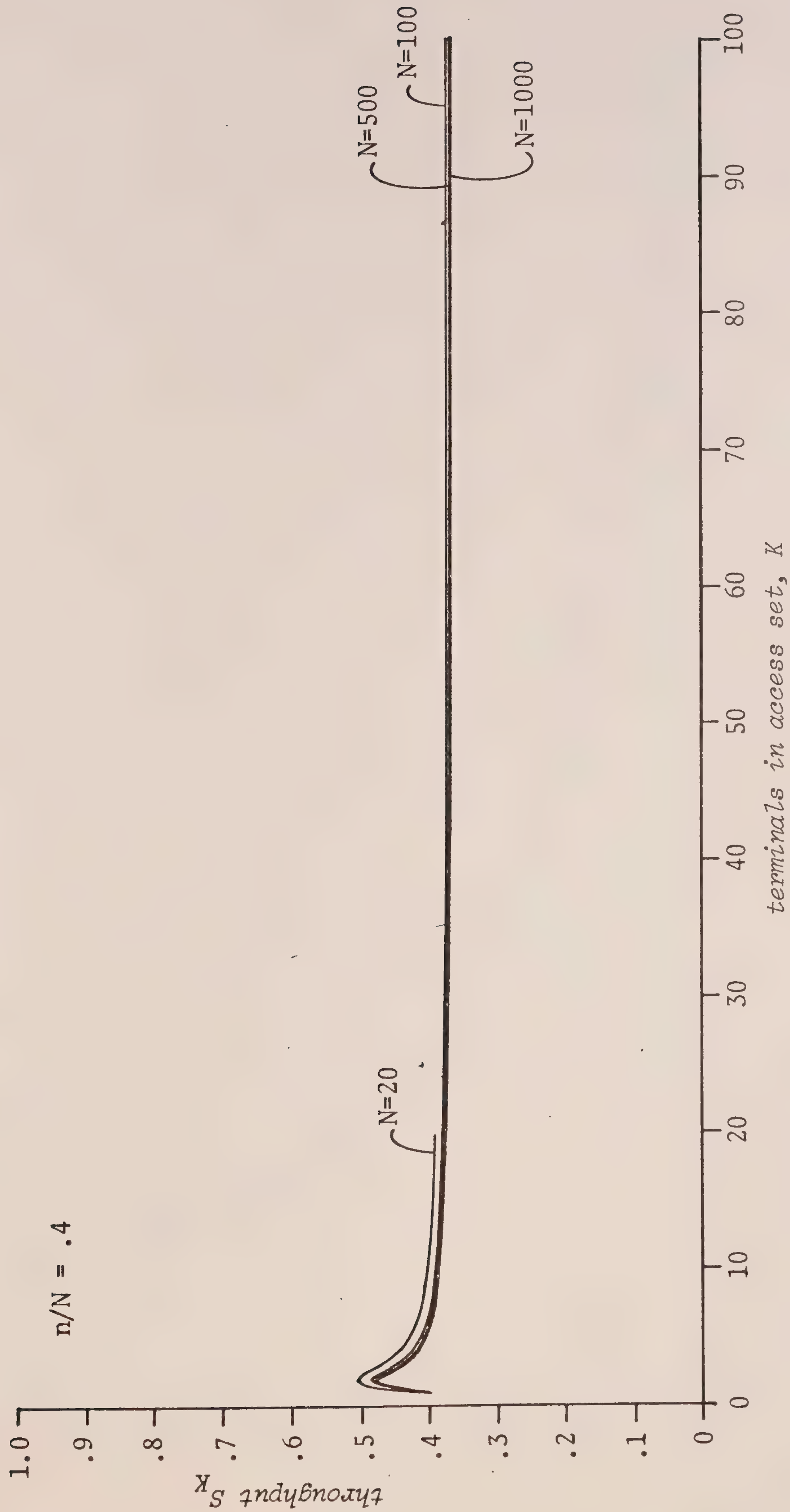
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
throughput S_K
n/N = .4
N=20
N=500
N=100
N=1000
0
10
20
30
40
50
60
70
80
90
100
terminals in access set, K

Figure 3

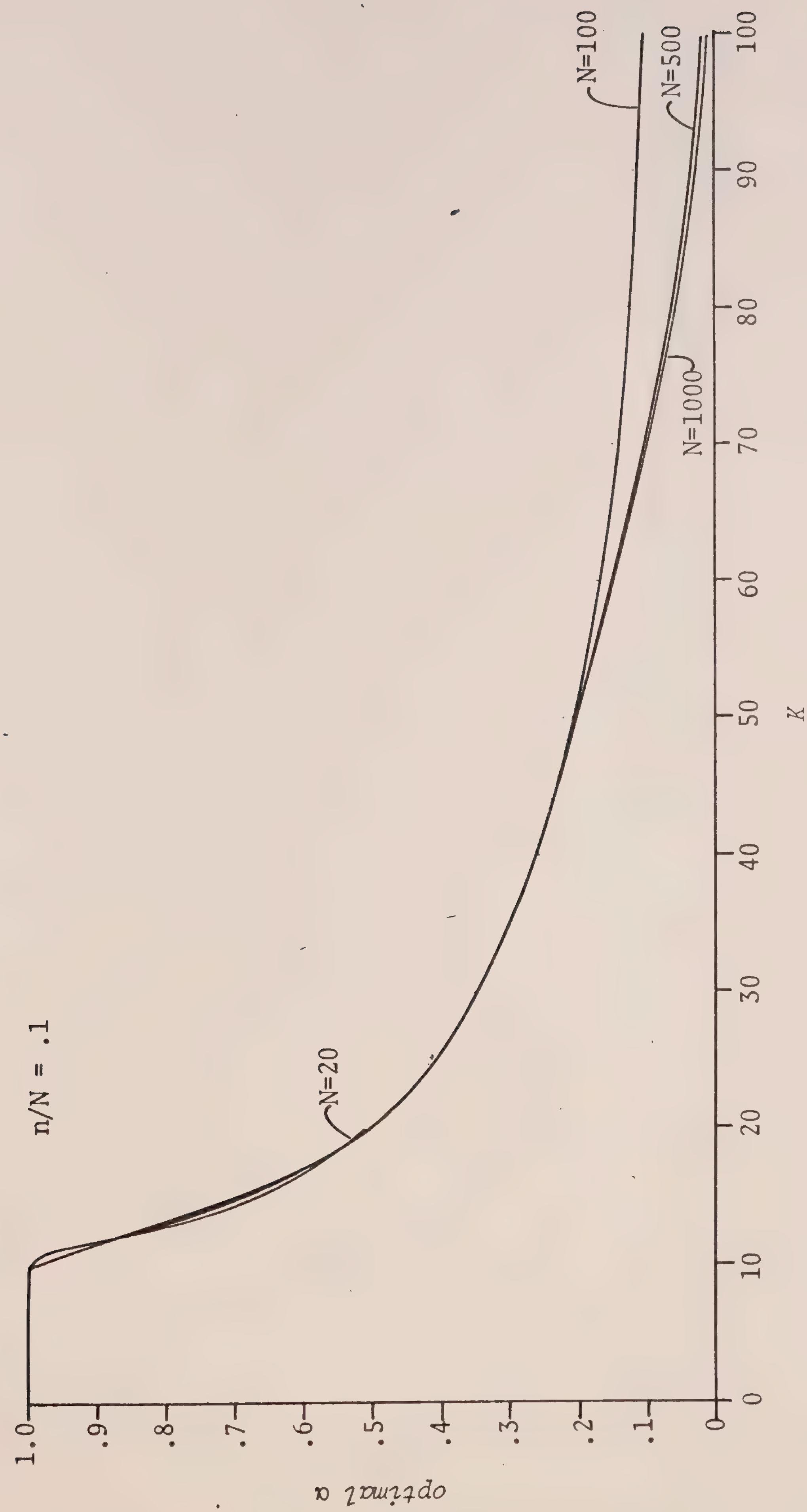
n/N = .1
N=20
N=100
N=500
N=1000
optimal α
K
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
10
20
30
40
50
60
70
80
90
100

Figure 4

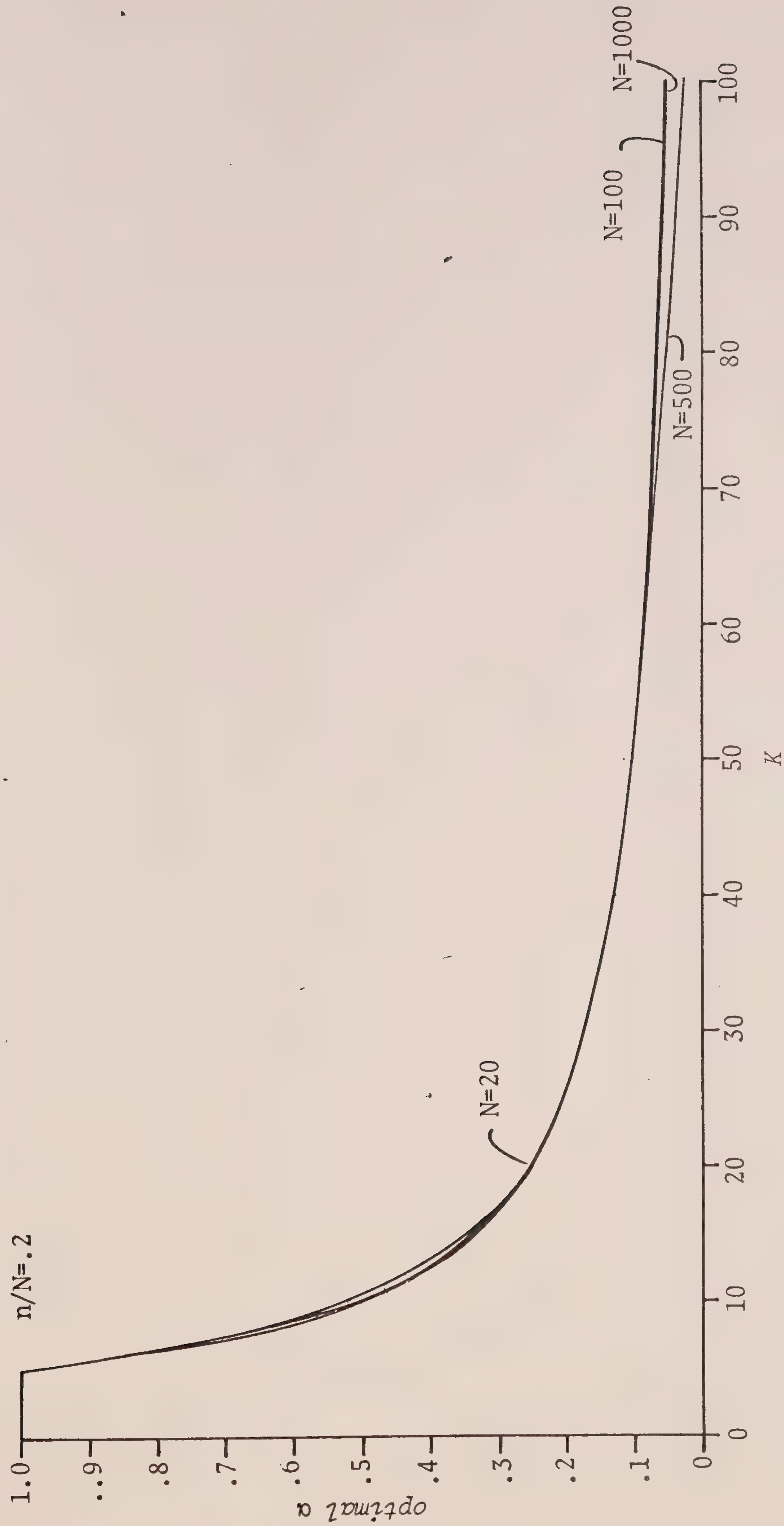
n/N=.2
N=20
N=100
N=500
N=1000
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
optimal α
0
10
20
30
40
50
60
70
80
90
100
K

Figure 5

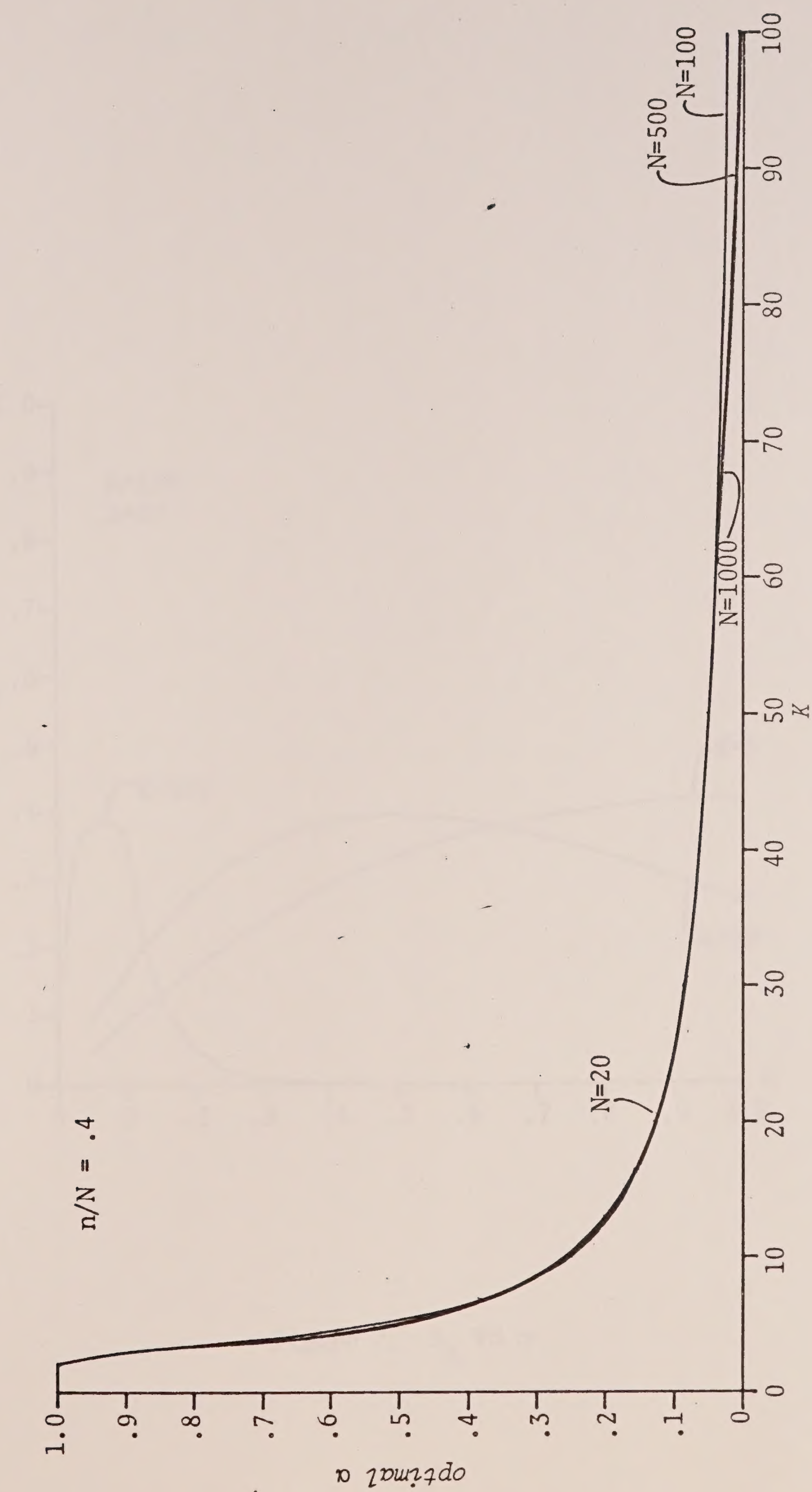
n/N = .4
optimal α
K
N=20
N=1000
N=500
N=100
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100

Figure 6

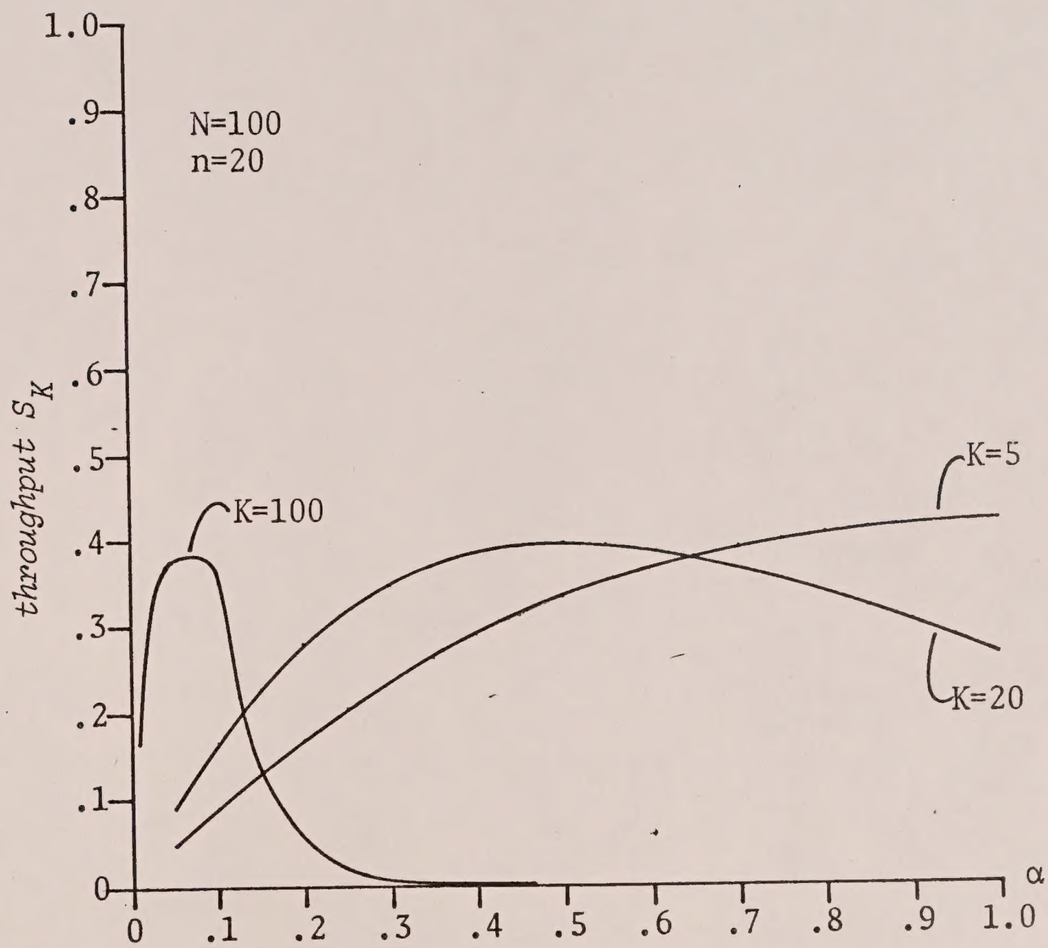

Figure 7. S_K VS α